DERIVATION OF AN UPPER BOUND OF THE CONSTANT IN THE ERROR BOUND FOR A NEAR BEST M-TERM APPROXIMATION

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Abstract. In [4], Temlyakov provides an error bound for a near best m-term approximation of a function $g \in L^p([0,1]^d)$, $1 < p < \infty$, $d \in \mathbb{N}$, using a basis L^p-equivalent to the Haar system $\mathcal{H}$. The bound includes a constant $C(p)$ that is not given explicitly. The goal of this paper is to find an upper bound of the constant for the Haar system $\mathcal{H}$, following the proof in [4].

1. Determining the constant in the one-dimensional case

Let $\mathcal{H} := \{H_I\}_I$ be the Haar basis in $L^p[0,1]$ indexed by dyadic intervals $I = [(j-1)2^{-n}, j2^{-n})$, $j = 1, ..., 2^n$, $n = 0, 1, ...$ and $I = [0,1]$ with

$$\begin{aligned} H_{[0,1]}(x) &= 1 \quad \text{for } x \in [0,1), \\ H_{[(j-1)2^{-n}, j2^{-n})}(x) &= \begin{cases} 2^{n/2}, & x \in [(j-1)2^{-n}, (j-\frac{1}{2})2^{-n}), \\ -2^{-n/2}, & x \in [(j-\frac{1}{2})2^{-n}, j2^{-n}), \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let

$$f = \sum_I c_I(f) H_I,$$

where

$$c_I(f) := (f, H_I) = \int_0^1 f(x) H_I(x) dx,$$

and denote

$$c_I(f,p) := \|c_I(f) H_I\|_p.$$

Then $c_I(f,p) \to 0$ as $|I| \to 0$.

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Denote by Λ_m a set of m dyadic intervals I such that

$$\min_{I\in\Lambda_m} c_I(f,p) \geq \max_{J\notin\Lambda_m} c_J(f,p).$$

This means that Λ_m contains the m largest values of $c_I(f,p)$ where I runs through all dyadic intervals. Then we define the Greedy algorithm $G_m^p(\cdot,\mathcal{H})$ as

$$G_m^p(f,\mathcal{H}) := \sum_{I\in\Lambda_m} c_I(f)H_I.$$

The following theorem provides an error bound for the approximation of a function $f \in L^p[0,1]$ by the Greedy algorithm $G_m^p(\cdot,\mathcal{H})$:

Theorem 1.1. *Let $1 < p < \infty$. Then for any $g \in L^p[0,1]$, we have*

$$\|g - G_m^p(g,\mathcal{H})\|_p \leq \left(2 + \frac{1}{\left(1-\left(\frac{1}{2}\right)^{1/p}\right)^2}\right) \cdot \left(\max\left(p, \frac{p}{p-1}\right) - 1\right)^2 \cdot \sigma_m(g)_p.$$

Proof. The Littlewood-Paley theorem for the Haar system gives for $1 < p < \infty$

$$C_3(p)\left\|\left(\sum_I |c_I(g)H_I|^2\right)^{\frac{1}{2}}\right\|_p \leq ||g||_p \leq C_4(p)\left\|\left(\sum_I |c_I(g)H_I|^2\right)^{\frac{1}{2}}\right\|_p. \tag{1}$$

In case of g being a martingale, explicit formulas for these constants are known (cf. [1]). In Lemma 1.6, page 8, it is shown that the Haar series

$$g = \sum_I c_I(g)H_I$$

is in fact a (conditionally symmetric) martingale.

Thus, taking the constants in [1], page 87, we have

$$C_3(p) = \frac{1}{\max\left(p, \frac{p}{p-1}\right) - 1} \quad \text{and} \quad C_4(p) = \max\left(p, \frac{p}{p-1}\right) - 1.$$

Let T_m be an m-term Haar polynomial of best m-term approximation to g in $L^p[0,1]$:

$$T_m = \sum_{I\in\Lambda} a_I H_I, \quad |\Lambda| = m.$$

For any finite set Q of dyadic intervals we denote by S_Q the projector

$$S_Q(f) := \sum_{I\in Q} c_I(f)H_I.$$

With these definitions, one can derive the following inequality:

$$\begin{aligned}
\|g - S_\Lambda(g)\|_p &= \|g - T_m - S_\Lambda(g - T_m)\|_p \\
&\le \|Id - S_\Lambda\|_{p\to p}\sigma_m(g)_p \\
&\le C_4(p)C_3(p)^{-1}\sigma_m(g)_p,
\end{aligned}$$

where Id denotes the identical operator. The last inequality holds since

$$\|g\|_p \le 1$$

implies by the Littlewood-Paley theorem (cf. (1)) that

$$\left\| \left(\sum_I |c_I(g)H_I|^2 \right)^{\frac{1}{2}} \right\|_p \le C_3(p)^{-1},$$

so that by again applying (1) we get

$$\begin{aligned}
\|(Id - S_\Lambda)(g)\|_p &= \left\| \sum_{I\notin\Lambda} c_I(g)H_I \right\|_p \\
&\le C_4(p) \left\| \left(\sum_{I\notin\Lambda} |c_I(g)H_I|^2 \right)^{\frac{1}{2}} \right\|_p \\
&\le C_4(p) \left\| \left(\sum_{I} |c_I(g)H_I|^2 \right)^{\frac{1}{2}} \right\|_p \\
&\le C_4(p)C_3(p)^{-1}.
\end{aligned}$$

With $C_3(p)$ and $C_4(p)$ given above we have

$$\|g - S_\Lambda(g)\|_p \le \left(\max\left(p, \frac{p}{p-1}\right) - 1 \right)^2 \cdot \sigma_m(g)_p. \tag{2}$$

Since

$$G_m^p(g) = S_{\Lambda_m}(g),$$

we have

$$
\begin{aligned}
& \|g - G_m^p(g)\|_p \\
\leq \; & \|g - S_\Lambda(g)\|_p + \|S_\Lambda(g) - S_{\Lambda_m}(g)\|_p \\
\overset{(2)}{\leq} \; & \left(\max\left(p, \frac{p}{p-1}\right) - 1 \right)^2 \cdot \sigma_m(g)_p + \|S_\Lambda(g) - S_{\Lambda_m}(g)\|_p. \qquad (3)
\end{aligned}
$$

It remains to estimate $\|S_\Lambda(g) - S_{\Lambda_m}(g)\|_p$ appropriately:

$$
\begin{aligned}
\|S_\Lambda(g) - S_{\Lambda_m}(g)\|_p &= \|S_{\Lambda \setminus \Lambda_m}(g) - S_{\Lambda_m \setminus \Lambda}(g)\|_p \\
&\leq \|S_{\Lambda \setminus \Lambda_m}(g)\|_p + \|S_{\Lambda_m \setminus \Lambda}(g)\|_p. \qquad (4)
\end{aligned}
$$

The second term in the last expression can be estimated by

$$
\begin{aligned}
\|S_{\Lambda_m \setminus \Lambda}(g)\|_p &= \| \left(Id - S_{(\Lambda_m \setminus \Lambda)^C} \right) (g) \|_p \\
&= \|g - T_m - S_{\Lambda \bigcup \Lambda_m^C}(g - T_m)\|_p \\
&\leq \|Id - S_{\Lambda \bigcup \Lambda_m^C}\|_{p \to p} \sigma_m(g)_p \\
&\leq \frac{C_4(p)}{C_3(p)} \sigma_m(g)_p \\
&= \left(\max\left(p, \frac{p}{p-1}\right) - 1 \right)^2 \cdot \sigma_m(g)_p. \qquad (5)
\end{aligned}
$$

Furthermore

$$
\|S_{\Lambda \setminus \Lambda_m}(g)\|_p \leq \frac{1}{\left(1 - \left(\frac{1}{2}\right)^{1/p}\right)^2} \cdot \|S_{\Lambda_m \setminus \Lambda}(g)\|_p \qquad (6)
$$

which will be derived in the following lemmas (Lemma 1.2 - 1.5).

Combining (3)–(6), we get

$$
\|g - G_m^p(g)\|_p \leq \left(2 + \frac{1}{\left(1 - \left(\frac{1}{2}\right)^{1/p}\right)^2} \right) \cdot \left(\max\left(p, \frac{p}{p-1}\right) - 1 \right)^2 \cdot \sigma_m(g)_p.
$$

□

Lemma 1.2. *Let $n_1 < n_2 < \cdots < n_s$ be integers and let $E_j \subset [0,1]$ be measurable sets, $j = 1, ..., s$. Then for any $0 < q < \infty$ we have*

$$\int_0^1 \left(\sum_{j=1}^s 2^{n_j/q}\chi_{E_j}(x)\right)^q dx \le \left(\frac{1}{1-\left(\frac{1}{2}\right)^{1/q}}\right)^q \cdot \sum_{j=1}^s 2^{n_j}|E_j|.$$

where $\chi_I(\cdot)$ is the characteristic function of the interval I:

$$\chi_I(x) = \begin{cases} 1, & x \in I, \\ 0, & x \notin I. \end{cases}$$

Proof. Denote

$$F(x) := \sum_{j=1}^s 2^{n_j/q}\chi_{E_j}(x)$$

and estimate it on the sets

$$E_l^- := E_l \setminus \bigcup_{k=l+1}^s E_k, \quad l = 1, ..., s-1; \quad E_s^- := E_s.$$

We have for $x \in E_l^-$

$$\begin{aligned} F(x) &\le \sum_{j=1}^l 2^{n_j/q} \\ &= 2^{n_l/q}\left(\frac{2^{n_1/q}}{2^{n_l/q}} + \cdots + 1\right) \\ &\le 2^{n_l/q}\sum_{i=0}^\infty \left(\frac{1}{2^{1/q}}\right)^i \\ &= 2^{n_l/q}\frac{1}{1-\left(\frac{1}{2}\right)^{1/q}}. \end{aligned}$$

Therefore,

$$\int_0^1 F(x)^q dx \le \left(\frac{1}{1-\left(\frac{1}{2}\right)^{1/q}}\right)^q \sum_{l=1}^s 2^{n_l}|E_l^-| \le \left(\frac{1}{1-\left(\frac{1}{2}\right)^{1/q}}\right)^q \sum_{l=1}^s 2^{n_l}|E_l|,$$

which proves the lemma. □

Lemma 1.3. *Consider*

$$f = \sum_{I \in Q} c_I H_I, \quad |Q| = N.$$

Let $1 \le p < \infty$. *Assume that*

$$\|c_I H_I\|_p \le 1, \quad I \in Q. \tag{7}$$

Then

$$\|f\|_p \le \frac{1}{1 - \left(\frac{1}{2}\right)^{1/p}} N^{1/p}.$$

Proof. Denote by $n_1 < n_2 < \cdots < n_s$ all integers such that there is $I \in Q$ with $|I| = 2^{-n_j}$. Introduce the sets

$$E_j := \bigcup_{I \in Q : |I| = 2^{-n_j}} I.$$

Then the number N of elements in Q can be written in the form

$$N = \sum_{j=1}^{s} |E_j| 2^{n_j}.$$

Furthermore, we have

$$\|c_I H_I\|_p = |c_I| |I|^{1/p - 1/2}.$$

The assumption (7) implies $|c_I| \le |I|^{1/2 - 1/p}$. Next, we have

$$\|f\|_p \le \left\| \sum_{I \in Q} |c_I H_I| \right\|_p \le \left\| \sum_{I \in Q} |I|^{-1/p} \chi_I(x) \right\|_p .$$

The right hand side of this inequality cna be rewritten as

$$Y := \left(\int_0^1 \left(\sum_{j=1}^{s} 2^{n_j/p} \chi_{E_j}(x) \right)^p dx \right)^{1/p} .$$

Applying Lemma 1.2 with $q = p$, we get

$$\|f\|_p \le Y \le \frac{1}{1 - (1/2)^{1/p}} \left(\sum_{j=1}^{s} |E_j| 2^{n_j} \right)^{1/p} = \frac{1}{1 - (1/2)^{1/p}} N^{1/p}.$$

□

Lemma 1.4. *Consider*

$$f = \sum_{I \in Q} c_I H_I, \quad |Q| = N.$$

Let $1 \le p < \infty$. *Assume*

$$\|c_I H_I\|_p \ge 1, \quad I \in Q.$$

Then

$$\|f\|_p \geq \left(1 - \left(\frac{1}{2}\right)^{1/p}\right) N^{1/p}.$$

Proof. Define

$$u := \sum_{I \in Q} \overline{c}_I |c_I|^{-1} |I|^{1/p-1/2} H_I,$$

where the bar means complex conjugate number. Then for $p' = \frac{p}{p-1}$ we have

$$\|\overline{c}_I |c_I|^{-1} |I|^{1/p-1/2} H_I\|_{p'} = 1$$

and, by Lemma 1.3

$$\|u\|_{p'} \leq \frac{1}{1 - \left(\frac{1}{2}\right)^{1/p}} N^{1/p'}.$$

Consider (f, u). We have on the one hand

$$(f, u) = \sum_{I \in Q} |c_I| |I|^{1/p-1/2} = \sum_{I \in Q} \|c_I H_I\|_p \geq N,$$

and on the other hand

$$(f, u) \leq \|f\|_p \|u\|_{p'},$$

so that

$$N \leq (f, u) \leq \|f\|_p \|u\|_{p'} \leq \|f\|_p \frac{1}{1 - \left(\frac{1}{2}\right)^{1/p}} N^{1/p'}$$

which implies

$$\|f\|_p \geq \left(1 - \left(\frac{1}{2}\right)^{1/p}\right) N^{1/p}.$$

□

Lemma 1.5. *Let $1 < p < \infty$. Then for any $g \in L^p[0,1]$ we have*

$$\|S_{\Lambda \setminus \Lambda_m}(g)\|_p \leq \frac{1}{\left(1 - \left(\frac{1}{2}\right)^{1/p}\right)^2} \cdot \|S_{\Lambda_m \setminus \Lambda}(g)\|_p.$$

Proof. Denote

$$A := \max_{I \in \Lambda \setminus \Lambda_m} \|c_I(g) H_I\|_p \quad \text{and} \quad B := \min_{I \in \Lambda_m \setminus \Lambda} \|c_I(g) H_I\|_p.$$

Then by the definition of Λ_m we have

$$B \geq A.$$

Using Lemma 1.3, we get

$$\|S_{\Lambda\setminus\Lambda_m}(g)\|_p \le A \cdot \frac{1}{1-\left(\frac{1}{2}\right)^{1/p}} \cdot |\Lambda \setminus \Lambda_m|^{1/p} \le B \cdot \frac{1}{1-\left(\frac{1}{2}\right)^{1/p}} \cdot |\Lambda \setminus \Lambda_m|^{1/p}. \tag{8}$$

Using Lemma 1.4, we get

$$\|S_{\Lambda_m\setminus\Lambda}(g)\|_p \ge B \cdot \left(1-\left(\frac{1}{2}\right)^{1/p}\right) \cdot |\Lambda_m \setminus \Lambda|^{1/p}$$

so that

$$|\Lambda_m \setminus \Lambda|^{1/p} \le \frac{1}{B \cdot \left(1-\left(\frac{1}{2}\right)^{1/p}\right)} \|S_{\Lambda_m\setminus\Lambda}(g)\|_p. \tag{9}$$

Since $|\Lambda| = |\Lambda_m| = m$, we have $|\Lambda_m \setminus \Lambda| = |\Lambda \setminus \Lambda_m|$ and finally get

$$\begin{aligned} \|S_{\Lambda\setminus\Lambda_m}(g)\|_p &\overset{(8)}{\le} B \cdot \frac{1}{1-\left(\frac{1}{2}\right)^{1/p}} \cdot |\Lambda \setminus \Lambda_m|^{1/p} \\ &\overset{(9)}{\le} \frac{1}{\left(1-\left(\frac{1}{2}\right)^{1/p}\right)^2} \|S_{\Lambda_m\setminus\Lambda}(g)\|_p. \end{aligned}$$

□

Lemma 1.6. *Let $f \in L^p[0,1]$. Then the Haar series*

$$g = \sum_I c_I(g) H_I$$

is a conditionally symmetric martingale.

Proof. First, we give a definition of a conditionally symmetric martingale (cf. [5] and [6]).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a nondecreasing sequence of σ-fields

$$\{\Omega, \phi\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}.$$

Let H be a real or complex Hilbert space with norm $|\cdot|$. A sequence of H-valued strongly integrable functions $(f_n)_{n\ge 1}$ is a martingale if for each $n \ge 1$, f_n is strongly measurable relative to $\mathcal{F}_n$, and for $n \ge 2$,

$$\mathbb{E}(d_n|\mathcal{F}_{n-1}) = 0 \quad \text{a.e.}$$

Here the difference sequence $(d_n)_{n\geq 1}$ is defined by $f_n = \sum_{i=1}^n d_i$, $n \geq 1$. In the following, we also call the limit $f = \sum_{i=1}^{\infty} d_n$ martingale if the corresponding sequence $(f_n)_{n\geq 1}$ is a martingale.

A martingale is called conditionally symmetric if d_{n+1} and $-d_{n+1}$ have the same conditional distribution given $d_1, ..., d_n$.

We can write the Haar series as

$$g = (f,\varphi)\varphi + \sum_{k=0}^{\infty} \sum_{2^k \leq j \leq 2^{k+1}-1} (g, \Psi_{j,k})\Psi_{j,k},$$

where

$$\begin{aligned}
\varphi(x) &= \begin{cases} 1, & x \in [0,1), \\ 0, & \text{otherwise}, \end{cases} \\
\Psi_{0,0}(x) &= \varphi(2x) - \varphi(2x-1), \\
\Psi_{j,k}(x) &= 2^{k/2} \cdot \Psi_{0,0}(2^k x - j).
\end{aligned}$$

Consider the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ defined by

$$\begin{aligned}
\Omega &= [0,1], \\
\mathcal{F} &= \mathcal{B}([0,1]), \\
\mathbb{P}(A) &= |A|, \quad A \in \mathcal{F},
\end{aligned}$$

and the sequence of σ-fields

$$\begin{aligned}
\{\Omega, \phi\} &= \quad \{\Omega, \emptyset\} \\
&\subset \sigma(\varphi) \\
&\subset \sigma(\varphi, \Psi_{0,0}) \\
&\subset \sigma(\varphi, \Psi_{0,0}, \Psi_{2,1}) \\
&\subset \sigma(\varphi, \Psi_{0,0}, \Psi_{2,1}, \Psi_{3,1}) \subset \cdots \\
&\subset \sigma(\varphi, \cdots, \Psi_{2^k,k}, \cdots, \Psi_{2^{k+1}-1,k}, \Psi_{2^{k+1},k+1}) \subset \cdots \\
&\subset \mathcal{F}.
\end{aligned}$$

We define

$$d_0 = (g,\varphi)\varphi, \quad d_1 = (g, \Psi_{0,0})\Psi_{0,0}, \quad d_2 = (g, \Psi_{2,1})\Psi_{2,1}, \cdots$$

and

$$c_0 = (g, \varphi), \quad c_1 = (g, \Psi_{0,0}), \quad c_2 = (g, \Psi_{2,1}), \cdots$$

where the indices of $(f, \Psi_{\cdot,\cdot})\Psi_{\cdot,\cdot}$ and $(f, \Psi_{\cdot,\cdot})$ increase as in the definition of the sequence of σ-fields.

For each fixed $n \in \mathbb{N}_0$ and each $i = 0, \cdots, n$, each of the sets $\{x : d_i(x) = c_i\}$, $\{x : d_i(x) = -c_i\}$, and $\{x : d_i(x) = 0\}$ is either a superset of the support of d_{n+1} or each of the sets and the support d_{n+1} are disjoint. This implies that

$$\begin{aligned} & \mathbb{P}(d_{n+1} = c_{n+1} | d_i = j_i, \ \ i \in \{1, ..., n\}), \ \ j_i \in \{c_i, -c_i, 0\}) \\ = \ & \mathbb{P}(d_{n+1} = -c_{n+1} | d_i = j_i, \ \ i \in \{1, ..., n\}), \ \ j_i \in \{c_i, -c_i, 0\}) \end{aligned}$$

so that the conditional distribution of d_{n+1} and $-d_{n+1}$ is the same given $d_1, ..., d_n$. Furthermore, we have $\mathbb{E}(d_{n+1} | \mathcal{F}_n) = 0$. $\square$

2. Extension of the calculation to the multidimensional case

A very common way to extend the Haar basis to $[0,1]^d$ is given by the following construction (cf. [2]). Let E denote the collection of nonzero vertices of $[0,1]^d$. For each $e \in E$, we define the multivariate functions

$$\Psi^e(x_1, \cdots, x_d) := \Psi^{e_1}(x_1) \cdots \Psi^{e_d}(x_d),$$

where $\Psi^0(x) = \varphi(x)$, $\Psi^1(x) = \Psi_{0,0}(x)$. Furthermore, let

$$\Psi^e_{j,k}(x) = 2^{kd/2} \cdot \Psi^e(2^k x - j), \quad k \geq 0, \quad 2^k \leq j_i \leq 2^{k+1} - 1, \quad i = 1, ..., d$$

and

$$\Psi^*(x) = 1, \quad x \in [0,1]^d.$$

Then the collection of functions Ψ^*, $\Psi^e_{j,k}$, $e \in E$, $k \geq 0$, $2^k \leq j_i \leq 2^{k+1} - 1$, $i = 1, \cdots, d$ forms a basis for $L^p[0,1]^d$.

By considering the set $\mathcal{D}$ of dyadic cubes I which form the supports of the functions Ψ^*, $\Psi^e_{j,k}$ and exchanging the notation of Ψ^*, $\Psi^e_{j,k}$ to H_I, we can also write the multivariate Haar basis as

$$\mathcal{H} = \{H_I\}_{I \in \mathcal{D}}.$$

Lemma 2.1. *Consider $f \in L^p[0,1]^d$ with corresponding Haar series*

$$f = (f, \Psi^*)\Psi^* + \sum_{e\in E}\sum_{k=0}^{\infty}\sum_{\substack{2^k \le j_i \le 2^{k+1}-1\\ i=1,\cdots,d}} (f, \Psi^e_{j,k})\Psi^e_{j,k}.$$

Then the inner double sum

$$\sum_{k=0}^{\infty}\sum_{\substack{2^k \le j_i \le 2^{k+1}-1\\ i=1,\cdots,d}} (f, \Psi^e_{j,k})\Psi^e_{j,k}$$

forms a conditionally symmetric martingale on $[0,1]^d$ for each fixed $e \in E$, but so does not the Haar series itself.

Proof. First we show that the Haar series itself does not form a conditionally symmetric martingale.

Let us assume that $d = 2$ and remark that the proof goes analogously for $d > 2$. We have

$$\begin{aligned}
\Psi^{(0,1)}_{(0,0),0}(x_1,x_2) &= \begin{cases} 1, & (x_1,x_2) \in [0,1]\times[0,\frac{1}{2}],\\ -1, & (x_1,x_2) \in [0,1]\times(\frac{1}{2},1],\\ 0, & \text{otherwise,}\end{cases}\\
\Psi^{(1,0)}_{(0,0),0}(x_1,x_2) &= \begin{cases} 1, & (x_1,x_2) \in [0,\frac{1}{2}]\times[0,1],\\ -1, & (x_1,x_2) \in (\frac{1}{2},1]\times[0,1],\\ 0, & \text{otherwise,}\end{cases}\\
\Psi^{(1,1)}_{(0,0),0}(x_1,x_2) &= \begin{cases} 1, & (x_1,x_2) \in [0,\frac{1}{2}]\times[0,\frac{1}{2}] \bigcup (\frac{1}{2},1]\times(\frac{1}{2},1],\\ -1, & (x_1,x_2) \in (\frac{1}{2},1]\times[0,\frac{1}{2}] \bigcup [0,\frac{1}{2}]\times(\frac{1}{2},1],\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}$$

Therefore, the functions $\Psi^{(0,1)}_{(0,0),0}$, $\Psi^{(1,0)}_{(0,0),0}$, and $\Psi^{(1,1)}_{(0,0),0}$ can be represented as in Figure 1.

Thus

$$\begin{aligned}
\mathbb{P}(\Psi^{(1,1)}_{(0,0),0} = 1 | \Psi^{(0,1)}_{(0,0),0} = 1, \Psi^{(1,0)}_{(0,0),0} = 1) &= 1,\\
\mathbb{P}(\Psi^{(0,1)}_{(0,0),0} = 1 | \Psi^{(1,0)}_{(0,0),0} = 1, \Psi^{(1,1)}_{(0,0),0} = 1) &= 1,\\
\mathbb{P}(\Psi^{(1,0)}_{(0,0),0} = 1 | \Psi^{(0,1)}_{(0,0),0} = 1, \Psi^{(1,1)}_{(0,0),0} = 1) &= 1,
\end{aligned}$$

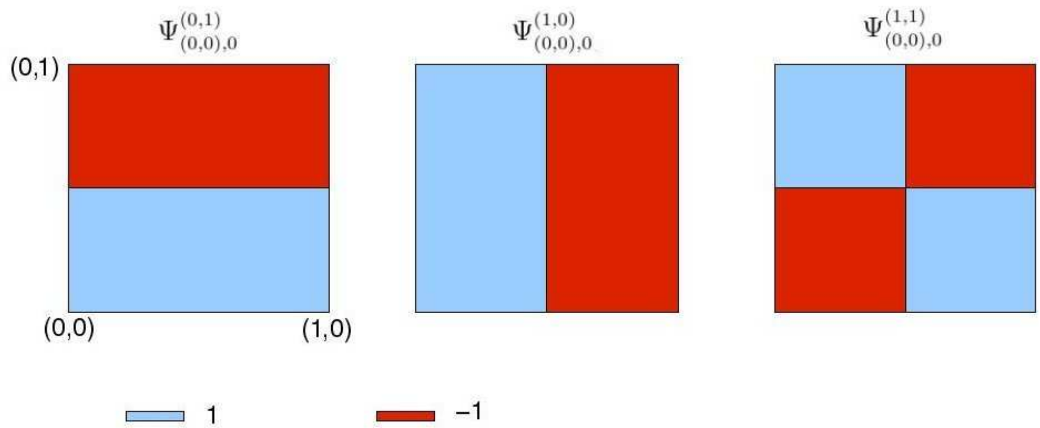

FIGURE 1. The functions $\Psi^{(0,1)}_{(0,0),0}$, $\Psi^{(1,0)}_{(0,0),0}$, and $\Psi^{(1,1)}_{(0,0),0}$.

which implies that

$$\begin{aligned}
\mathbb{E}\left(\Psi^{(1,1)}_{(0,0),0}\,\middle|\,\Psi^{(0,1)}_{(0,0),0}=1,\Psi^{(1,0)}_{(0,0),0}=1\right) &= 1,\\
\mathbb{E}\left(\Psi^{(0,1)}_{(0,0),0}\,\middle|\,\Psi^{(1,0)}_{(0,0),0}=1,\Psi^{(1,1)}_{(0,0),0}=1\right) &= 1,\\
\mathbb{E}\left(\Psi^{(1,0)}_{(0,0),0}\,\middle|\,\Psi^{(0,1)}_{(0,0),0}=1,\Psi^{(1,1)}_{(0,0),0}=1\right) &= 1.
\end{aligned}$$

Therefore the multiparameter Haar series cannot a martingale.

Let us now consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\begin{aligned}
\Omega &= [0,1]^d,\\
\mathcal{F} &= \mathcal{B}([0,1]^d),\\
\mathbb{P}(A) &= |A|, \quad A \subset \mathcal{F},
\end{aligned}$$

with the sequence of σ-fields

$$\begin{aligned}\{\Omega, \phi\} \quad = \quad & \sigma(\Psi^e_{(0,0),0}) \\ & \subset \sigma(\Psi^e_{(0,0),0}, \Psi^e_{(2,2),1}) \subset \cdots \\ & \subset \sigma(\Psi^e_{(0,0),0}, \cdots, \Psi^e_{(3,2),1}, \Psi^e_{(2,3),1}, \Psi^e_{(3,3),1}) \subset \cdots \\ & \subset \sigma(\Psi^e_{(0,0),0}, \cdots, \Psi^e_{(3,3),1}, \Psi^e_{(4,4),2}, \Psi^e_{(5,4),2}, \Psi^e_{(6,4),2}, \Psi^e_{(7,4),2}) \subset \cdots \\ & \subset \sigma(\Psi^e_{(0,0),0}, \cdots, \Psi^e_{(7,4),2}, \Psi^e_{(4,5),2}, \Psi^e_{(5,5),2}, \Psi^e_{(6,5),2}) \subset \cdots \\ & \subset \mathcal{F}\end{aligned}$$

for each vertex $e \in E$. We denote this sequence of σ-fields by

$$\mathcal{F}_0 = \sigma(\Psi^e_{(0,0),0}), \quad \mathcal{F}_1 = \sigma(\Psi^e_{(0,0),0}, \Psi^e_{(2,2),1}), \quad \cdots .$$

Furthermore, for a fixed $e \in E$, we consider the partial sums of the inner sums of the Haar series

$$\sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} (f, \Psi^e_{j,k}) \Psi^e_{j,k} \tag{10}$$

and show that they form a conditionally symmetric martingale.

Let us denote the difference sequence $(d_n)_{n\geq 1}$ of (10) by

$$d_0 = (f, \Psi^e_{(0,0),0}) \Psi^e_{(0,0),0}, \quad d_1 = (f, \Psi^e_{(2,2),1}) \Psi^e_{(2,2),1}, , \quad \cdots .$$

The corresponding coefficients are denoted by

$$c_0 = (f, \Psi^e_{(0,0),0}), \quad c_1 = (f, \Psi^e_{(2,2),1}), \quad \cdots .$$

For a fixed k and $e \in E$, the support of the functions $\Psi^e_{j,k}$ is disjoint. Furthermore, for $l < k$ and $e \in E$, each of the sets $\{x : \Psi^e_{j,l}(x) = 1\}$, $\{x : \Psi^e_{j,l}(x) = -1\}$, and $\{x : \Psi^e_{j,l}(x) = 0\}$ is either a superset of the support of $\Psi^e_{j,k}$ or each of the sets and the support of $\Psi^e_{j,k}$ are disjoint.

This implies that $-d_{n+1}$ and d_{n+1} have the same conditional distribution given $d_1, ..., d_n$ and therefore

$$E(d_n | \mathcal{F}_{n-1}) = 0.$$

Thus, the partial sums of

$$\sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} (f, \Psi_{j,k}^e) \Psi_{j,k}^e$$

form a conditionally symmetric martingale. $\square$

It is clear that the series $(f, \Psi^*)\Psi^* + \sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} (f, \Psi_{j,k}^e) \Psi_{j,k}^e$ is also a martingale.

Theorem 2.2. *Let $1 < p \leq 2$. Then for any $g \in L^p[0,1]^d$ we have*

$$\begin{aligned} &\|g - G_m^p(g, \mathcal{H})\|_p \\ &\leq \left(2 + \frac{1}{\left(1 - \left(\frac{1}{2}\right)^{d/p}\right)^2}\right) \left(\left(2^d - 1\right)\left(\max\left(p, \frac{p}{p-1}\right) - 1\right)\right)^2 \sigma_m(g)_p. \end{aligned}$$

Proof. Using Lemma 2.1, we get an estimate for the upper bound in the Littlewood-Paley inequality by additionally applying the triangle inequality. Let $e^* \in E$. Then

$$\begin{aligned} &\left\| (g, \Psi^*)\Psi^* + \sum_{e \in E} \sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} (g, \Psi_{j,k}^e) \Psi_{j,k}^e \right\|_p \\ \leq &\left\| (g, \Psi^*)\Psi^* + \sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} (g, \Psi_{j,k}^{e^*}) \Psi_{j,k}^{e^*} \right\|_p \\ &+ \sum_{e \in E \setminus \{e^*\}} \left\| \sum_{k=0}^{\infty} \sum_{\substack{2^k \leq j_i \leq 2^{k+1}-1 \\ i=1,\cdots,d}} (g, \Psi_{j,k}^e) \Psi_{j,k}^e \right\|_p \\ \leq &\sum_{e \in E} C_4(p) \left\| \left(\sum_{I \in \mathcal{D}} |c_I(g) H_I|^2 \right)^{\frac{1}{2}} \right\|_p \\ = &\left(2^d - 1\right) C_4(p) \left\| \left(\sum_{I \in \mathcal{D}} |c_I(g) H_I|^2 \right)^{\frac{1}{2}} \right\|_p . \end{aligned} \tag{11}$$

We now apply the method of duality (cf. [3]) in order to determine the lower bound of the Littlewood-Paley inequality. The idea is to consider

$$S(g) := \left\| \left(\sum_{I \in \mathcal{D}} |c_I(g) H_I|^2 \right)^{\frac{1}{2}} \right\|_p$$

as an element of $L^p_{l^2(\mathcal{D})}$, that is

$$\varphi = \left\{ \sqrt{|c_I(g) H_I|^2} : I \in \mathcal{D} \right\}$$

is considered as a p-integrable function taking values in $l^2(\mathcal{D})$. Due to the Hahn-Banach theorem, the dual function $\gamma = \{\gamma_I(x) : I \in \mathcal{D}\} \in L^{p'}_{l^2(\mathcal{D})}$ is of norm one and satisfies

$$\|\varphi\|_{L^p_{l^2}} = (\varphi, \gamma) = \sum_{I \in \mathcal{D}} \sqrt{|c_I(g) H_I|^2} \int_I \gamma_I dy,$$

where p' is the conjugate index, i. e. $1/p + 1/p' = 1$. This implies that we can assume that γ_I is supported on I and constant on I since in the above formula, only the mean value of γ_I over I is important.

By defining the function

$$h := \sum_{I \in \mathcal{D}} \left(\gamma_I \sqrt{|I|} \right) H_I,$$

we have on the one hand

$$S(h) = \|\gamma\|_{l^2(\mathcal{D})}$$

and on the other hand

$$\begin{aligned}
\|S(g)\|_p = \|\varphi\|_{L^p_{l^2}} &= (\varphi, \gamma) \\
&= \sum_{I \in \mathcal{D}} c_I(g) \gamma_I \sqrt{|I|} \\
&= (g, h) \\
&\leq \|g\|_p \|h\|_{p'} \\
&\overset{(11)}{\leq} \|g\|_p \left(2^d - 1\right) C_4(p) \|S(h)\|_{p'} \\
&= \|g\|_p \left(2^d - 1\right) C_4(p) \left\| \|\gamma\|_{l^2(\mathcal{D})} \right\|_{p'} \\
&= \|g\|_p \left(2^d - 1\right) C_4(p).
\end{aligned}$$

so that in the multidimensional case, the Littlewood-Paley inequality reads

$$\frac{1}{C_4^*(p,d)}\left\|\left(\sum_I |c_I(g)H_I|^2\right)^{\frac{1}{2}}\right\|_p \leq ||g||_p \leq C_4^*(p,d)\left\|\left(\sum_I |c_I(g)H_I|^2\right)^{\frac{1}{2}}\right\|_p, \qquad (12)$$

where $C_4^*(p,d) = \left(2^d-1\right)C_4(p) = \left(2^d-1\right)\left(\max\left(p,\frac{p}{p-1}\right)-1\right)$.

Now, the remainder of the proof goes as for the univariate case. We note that

$$\begin{aligned} \|g - S_\Lambda(g)\|_p &\leq C_4^*(p,d)^2 \cdot \sigma_m(g)_p, \\ \|S_{\Lambda_m\setminus\Lambda}(g)\|_p &\leq C_4^*(p,d)^2 \cdot \sigma_m(g)_p, \end{aligned}$$

and

$$\|S_{\Lambda\setminus\Lambda_m}(g)\|_p \leq \frac{1}{\left(1-\left(\frac{1}{2}\right)^{d/p}\right)^2}\|S_{\Lambda_m\setminus\Lambda}(g)\|_p$$

which will be derived in the following lemmas.

Combining the last three inequalities, we get

$$\begin{aligned} & \|g - G_m^p(g)\|_p \\ \leq\ & \|g - S_\Lambda(g)\|_p + \|S_\Lambda(g) - S_{\Lambda_m}(g)\|_p \\ =\ & \|g - S_\Lambda(g)\|_p + \|S_{\Lambda\setminus\Lambda_m}(g) - S_{\Lambda_m\setminus\Lambda}(g)\|_p \\ \leq\ & \|g - S_\Lambda(g)\|_p + \|S_{\Lambda\setminus\Lambda_m}(g)\|_p + \|S_{\Lambda_m\setminus\Lambda}(g)\|_p \\ \leq\ & \left(2+\frac{1}{\left(1-\left(\frac{1}{2}\right)^{d/p}\right)^2}\right)\left(\left(2^d-1\right)\left(\max\left(p,\frac{p}{p-1}\right)-1\right)\right)^2\sigma_m(g)_p. \end{aligned}$$

□

Lemma 2.3. *Let $n_1 < n_2 < \cdots < n_s$ be integers and let $E_j \subset [0,1]^d$ be measurable sets, $j = 1, ..., s$. Then for any $0 < q < \infty$ we have*

$$\int_{[0,1]^d}\left(\sum_{j=1}^s 2^{n_j d/q}\chi_{E_j}(x)\right)^q dx \leq \left(\frac{1}{1-\left(\frac{1}{2}\right)^{d/q}}\right)^q \cdot \sum_{j=1}^s 2^{n_j d}|E_j|.$$

Proof. Denote

$$F(x) := \sum_{j=1}^s 2^{n_j d/q}\chi_{E_j}(x)$$

and estimate it on the sets

$$E_l^- := E_l \setminus \bigcup_{k=l+1}^{s} E_k, \quad l = 1, ..., s-1; \quad E_s^- := E_s.$$

We have for $x \in E_l^-$

$$\begin{aligned} F(x) &\le \sum_{j=1}^{l} 2^{n_j d/q} = 2^{n_l d/q}\left(\frac{2^{n_1 d/q}}{2^{n_l d/q}} + \cdots + 1\right) \\ &\le 2^{n_l d/q} \sum_{i=0}^{\infty} \left(\frac{1}{2^{d/q}}\right)^i = 2^{n_l d/q} \frac{1}{1-\left(\frac{1}{2}\right)^{d/q}}. \end{aligned}$$

Therefore,

$$\int_0^1 F(x)^q dx \le \left(\frac{1}{1-\left(\frac{1}{2}\right)^{d/q}}\right)^q \sum_{l=1}^{s} 2^{n_l d}|E_l^-| \le \left(\frac{1}{1-\left(\frac{1}{2}\right)^{d/q}}\right)^q \sum_{l=1}^{s} 2^{n_l d}|E_l|,$$

which proves the lemma. $\square$

Lemma 2.4. *Consider*

$$f = \sum_{I \in Q} c_I H_I, \quad |Q| = N.$$

Let $1 \le p < \infty$. *Assume that*

$$\|c_I H_I\|_p \le 1, \quad I \in Q. \tag{13}$$

Then

$$\|f\|_p \le \frac{1}{1-\left(\frac{1}{2}\right)^{d/p}} N^{1/p}.$$

Proof. Denote by $n_1 < n_2 < \cdots < n_s$ all integers such that there is $I \in Q$ with $|I| = 2^{-dn_j}$. Introduce the sets

$$E_j := \bigcup_{I \in Q : |I| = 2^{-dn_j}} I.$$

Then the number N of elements in Q can be written in the form

$$N = \sum_{j=1}^{s} |E_j| 2^{dn_j}.$$

Furthermore, we have for $|I| = 2^{-kd}$, $k = 0, 1, 2, \cdots$

$$\|c_i H_I\|_p = |c_I| \left(\int\limits_I \left|2^{dk/2}\right|^p dx \right)^{1/p} = |c_I| \cdot 2^{dk/2} \cdot 2^{-kd/p} = |c_I||I|^{1/p-1/2}.$$

The assumption (13) implies

$$|c_I| \leq |I|^{1/2-1/p}.$$

Next, we have

$$\|f\|_p \leq \left\| \sum_{I \in Q} |c_I H_I| \right\|_p \leq \left\| \sum_{I \in Q} |I|^{-1/p} \chi_I(x) \right\|_p .$$

The right hand side of this inequality can be rewritten as

$$Y := \left(\int_{[0,1]^d} \left(\sum_{j=1}^s 2^{dn_j/p} \chi_{E_j}(x) \right)^p dx \right)^{1/p} .$$

Applying Lemma 2.3 with $q = p$, we get

$$\|f\|_p \leq Y \leq \frac{1}{1 - \left(\frac{1}{2}\right)^{d/p}} \left(\sum_{j=1}^s |E_j| 2^{dn_j} \right)^{1/p} = \frac{1}{1 - \left(\frac{1}{2}\right)^{d/p}} N^{1/p}.$$

□

Lemma 2.5. *Consider*

$$f = \sum_{I \in Q} c_I H_I, \quad |Q| = N.$$

Let $1 \leq p < \infty$. Assume that

$$\|c_I H_I\|_p \geq 1, \quad I \in Q. \tag{14}$$

Then

$$\|f\|_p \geq \left(1 - \left(\frac{1}{2}\right)^{d/p} \right) N^{1/p}.$$

Proof. Define

$$u := \sum_{I \in Q} \overline{c}_I |c_I|^{-1} |I|^{1/p-1/2} H_I,$$

where the bar means complex conjugate number. Then for $p' = \frac{p}{p-1}$ we have

$$\|\overline{c}_I |c_I|^{-1} |I|^{1/p-1/2} H_I\|_{p'} = 1$$

and, by Lemma 2.3

$$\|u\|_{p'} \leq \frac{1}{1-\left(\frac{1}{2}\right)^{d/p}} N^{1/p'}.$$

Consider (f,u). We have on the one hand

$$(f,u) = \sum_{I \in Q} |c_I||I|^{1/p-1/2} = \sum_{I \in Q} \|c_I H_I\|_p \geq N,$$

and on the other hand

$$(f,u) \leq \|f\|_p \|u\|_{p'},$$

so that

$$N \leq (f,u) \leq \|f\|_p \|u\|_{p'} \leq \|f\|_p \frac{1}{1-\left(\frac{1}{2}\right)^{d/p}} N^{1/p'}$$

which implies

$$\|f\|_p \geq \left(1-\left(\frac{1}{2}\right)^{d/p}\right) N^{1/p}.$$

$\square$

Lemma 2.6. *Let $1 < p < \infty$. Then for any $g \in L^p[0,1]^d$ we have*

$$\|S_{\Lambda \setminus \Lambda_m}(g)\|_p \leq \frac{1}{\left(1-\left(\frac{1}{2}\right)^{d/p}\right)^2} \cdot \|S_{\Lambda_m \setminus \Lambda}(g)\|_p.$$

Proof. Denote

$$A := \max_{I \in \Lambda \setminus \Lambda_m} \|c_I(g) H_I\|_p \quad \text{and} \quad B := \min_{I \in \Lambda_m \setminus \Lambda} \|c_I(g) H_I\|_p.$$

Then by the definition of Λ_m we have

$$B \geq A.$$

Using Lemma 2.3, we get

$$\|S_{\Lambda \setminus \Lambda_m}(g)\|_p \leq A \cdot \frac{1}{1-\left(\frac{1}{2}\right)^{d/p}} \cdot |\Lambda \setminus \Lambda_m|^{1/p} \leq B \cdot \frac{1}{1-\left(\frac{1}{2}\right)^{d/p}} \cdot |\Lambda \setminus \Lambda_m|^{1/p}. \tag{15}$$

Using Lemma 2.4, we get

$$\|S_{\Lambda_m \setminus \Lambda}(g)\|_p \geq B \cdot \left(1-\left(\frac{1}{2}\right)^{d/p}\right) \cdot |\Lambda_m \setminus \Lambda|^{1/p}$$

so that

$$|\Lambda_m \setminus \Lambda|^{1/p} \leq \frac{1}{B \cdot \left(1 - \left(\frac{1}{2}\right)^{d/p}\right)} \|S_{\Lambda_m \setminus \Lambda}(g)\|_p. \tag{16}$$

Taking into account that $|\Lambda_m \setminus \Lambda| = |\Lambda \setminus \Lambda_m|$, we get

$$\|S_{\Lambda \setminus \Lambda_m}(g)\|_p \stackrel{(15)}{\leq} B \cdot \frac{1}{1-\left(\frac{1}{2}\right)^{d/p}} \cdot |\Lambda \setminus \Lambda_m|^{1/p} \stackrel{(16)}{\leq} \frac{1}{\left(1 - \left(\frac{1}{2}\right)^{d/p}\right)^2} \|S_{\Lambda_m \setminus \Lambda}(g)\|_p.$$

□

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